THE EQUIVALENCE PRINCIPLE

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Summary

In this chapter we survey some equivalence results. The starting point is the set of Walrasian allocations. We first show that a Walrasian allocation can be characterized by the property that it has strongly fair net trades. Then we consider atomless economies. An atomless economy formalizes the assumption that the economy consists of many small agents. In an atomless economy we present the Core, the Bargaining Set, and Value equivalence results.

We also examine large finite economies. We present a Core decentralization result and also a decentralization result for the Geanakoplos Bargaining Set. The Mas-Colell Bargaining Set does not lead to a convergence result in large finite economies.

Finally, we give a few examples of equivalence between the set of Walrasian equilibria in a finite economy and the set of Nash equilibria in suitably defined non-cooperative games.

1. Introduction

The starting point of this survey is a pure exchange economy with finitely many commodities and with private ownership of the initial endowments. In such an economy it is often assumed that a Walrasian market gives the trading possibilities for the consumers. A Walrasian market is the institution given by a price system. All consumers take the prices of the commodities as parametrically given and choose an optimal action given these prices. The prices defining the Walrasian market are set such that aggregate demands equal aggregate supplies. Much of economic theory is devoted to analyzing economies with Walrasian markets or variants of this model.

However, considering an economy with a Walrasian market does not justify the Walrasian institution. How can it be justified that the trading possibilities for the agents are defined by a price system and that agents take the price system as parametrically given? Game theory has been extremely useful in the search for an answer to this question.

Concepts from cooperative as well as non-cooperative game theory have been used to introduce new equilibrium concepts into economics. These equilibrium notions do not rest on the assumption that agents take the prices of commodities as given. Thus, one has been able to ask the question, whether some of these other equilibrium notions lead to an equivalence result in the following sense: An allocation of the commodities to the agents in the economy is an equilibrium state according to this new equilibrium concept if and only if there exists a price system p such that the allocation is an equilibrium allocation corresponding to the Walrasian market defined by the price system p. If an equivalence result obtains we have an endogenous explanation of the Walrasian institution.

For most of the equilibrium concepts used in game theory there is no assumption paralleling the assumption that the agents take the Walrasian market as given *a priori*. Clearly, if prices are always set such that demands equal supplies, then in a finite economy any agent shall be able to influence the price system. However, the implicit assumption is that agents behave as if their actions have no affect on the price system. Clearly, one may think, that if the economy consists of many small agents who act independently, then this implicit assumption is approximately satisfied. Aumann (1964) defined a continuum economy in which the agents were modeled as [0, 1] with the Lebesgue measure. In Aumann's model the assumption that an individual agent cannot influence the price system is endogenous and Aumann gave the first general equivalence theorem. He proves that an allocation can be obtained via a Walrasian market if and only if there is no group of consumers, which by using its own initial endowments can ensure that all its members are better of. This is Aumann's classical Core equivalence theorem.

Since Aumann's result, many other equivalence results have been obtained for economies with an atomless measure space of consumers. These results have very much enlarged our understanding of the foundation for the Walrasian market institution. Moreover, the attempts to analyze economies with infinitely many commodities have given new insights. Ostroy and Zame (1994) have pointed out that, when the commodity space is infinitely dimensional, an atomless measure space of agents is, in general, not enough to obtain results analogous with the equivalence results for economies with finitely many commodities.

Clearly, modeling the agents in an economy as an atomless measure space is an abstraction. Hence, a fundamental question is whether the equivalence results for atomless economies have analogies in economies with large, but finite numbers of agents. A strong result in this direction is the classical theorem by Debreu and Scarf (1963). They showed that the Core and set of Walrasian allocations become arbitrarily close when a finite economy is replicated sufficiently many times. However, Bewley (1973) showed that if one considers more general sequences of finite economies, one cannot, in general, hope for such a strong conclusion. This leads to a weaker question. Namely, whether for some of the game theoretical solution concepts, one will have that any equilibrium allocation can be approximately decentralized by a Walrasian market in large finite economies.

Searching for equivalence results has a parallel in classical welfare economics. For a long time, it has been known that allocations obtained via a Walrasian market are Pareto efficient. However, starting with a Pareto efficient allocation, a transfer of initial endowments among the agents is necessary if the allocation has to be obtained from a Walrasian market. This is the content of the classical First and Second welfare theorems, see Debreu (1959).

2. Notation and the Basic Model

For two vectors $x, y \in \mathbb{R}^{\ell}$, we use the notation $y \geq x$ if $y_h \geq x_h$ for all $h = 1, ..., \ell$; $y > x$ if $y_h \geq x_h$ for all $h = 1, ..., \ell$ and $y \neq x$; and $y >> x$ if $y_h > x_h$ for all $h = 1, ..., \ell$. We let $\Delta = \{p \in \mathbb{R}^{\ell}_{+} \mid \sum p_h = 1\}$ be the non-negative price simplex in $\mathbb{R}^{\ell}$. For a set S let $|S|$ denote the number of elements in S. $\mathbf{Z}_{+}$ denotes the non-negative integers. For $x \in \mathbb{R}$ we let $\|x\|$ denote the Euclidean norm of x.

We consider economies in which all consumers have the positive orthant $\mathbb{R}^{\ell}_{+}$ as *consumption sets*.

A *preference relation* $\succ$ on $\mathbb{R}^{\ell}_{+}$ is said to be *continuous* if the set $\{(x, y) \in \mathbb{R}^{\ell}_{+} \times \mathbb{R}^{\ell}_{+} \mid y \succ x\}$ is open relative to $\mathbb{R}^{\ell}_{+} \times \mathbb{R}^{\ell}_{+}$. The relation $\succ$ is irreflexive if $x \not\succ x$ for all $x \in \mathbb{R}^{\ell}_{+}$. It is *monotonic* if for all $x, y \in \mathbb{R}^{\ell}_{+}$ with $y > x$ we have $y \succ x$. A preference relation on $\mathbb{R}^{\ell}_{+}$ is said to be *transitive-monotonic* if $z \geq y$ and $y \succ x$ imply $z \succ x$ for all $x, y, z \in \mathbb{R}^{\ell}_{+}$. We let $\mathcal{P}_{mo}$ be the set of continuous, irreflexive, monotonic, and transitive-monotonic preference relations on $\mathbb{R}^{\ell}_{+}$. A preference relation $\succsim$ on $\mathbb{R}^{\ell}_{+}$ is complete if $y \succsim x$ or $x \succsim y$ for all $x, y \in \mathbb{R}^{\ell}_{+}$. The relation $\succsim$ is transitive if $z \succsim y$ and $y \succsim x$ imply $z \succsim x$ for all $x, y, z \in \mathbb{R}^{\ell}_{+}$. We say that $\succ \in \mathcal{P}_{mo}$ is derived from the complete and transitive preference relation $\succsim$ when $y \succ x$ if and only if $y \succsim x$ and $x \not\succsim y$. We let $\mathcal{P}^{*}_{mo} = \{\succ \in \mathcal{P}_{mo} \mid \succ$ is derived from a complete and transitive relation $\succsim\}$. A preference relation $\succ \in \mathcal{P}^{*}_{mo}$ is said to be *smooth* if the corresponding preference relation $\succsim$ can be represented by a strictly quasiconcave C^2 utility function $u : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ with positive Gaussian curvature u^c. (The function u is strictly quasiconcave, if $u(\lambda x + (1 - \lambda)y) >$

$\min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}^{\ell}_{+}$, $x \neq y$, and $\lambda \in (0, 1)$.)

A pure *exchange economy* with private ownership is a mapping $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}^{\ell}_{+} \times \mathcal{P}_{mo}$, where $a \mapsto \mathcal{E}(a) = (e(a), \succ_a)$. $\mathcal{A}$ is a σ-field of subsets of A. λ is a finite non-negative measure on $\mathcal{A}$. A is the set of consumers. An element $S \in \mathcal{A}$ is a *coalition* of consumers. A coalition S is said to be *non-null* if $\lambda(S) > 0$. We shall assume that the measure space is complete. Thus, all sets $S \subset A$ for which there exists a null set $T \in$ where $S \subset T$ are again in . The vector $e(a)$ is the initial endowment of consumer a and $\in$ is consumer a's preference relation on . We assume that $e : A \to$ is an integrable function with . Furthermore, we assume that is measurable in the sense that for any measurable functions $f, g : A \to$ we have $\{a \in A \mid f(a)\ g(a)\} \in$.

Consider a consumer a in the economy and a consumption plan $x \in$. Then we define a's *net trade* as $x - e(a)$. Since we have assumed that the consumers' consumption sets equal then the set of net trades which are individually feasible for a is $- \{e(a)\}$.

Definition 1

Let be an economy. An allocation *for the coalition S is an integrable function* $x : S \to$. *An* attainable allocation x for the coalition $S \in$ *is an allocation for S such that*

$x(a) \in$ *for a.a.* $a \in A$ *and* .

An attainable allocation x *is an allocation which is attainable for A. We let X() denote the allocations that are attainable in the economy* .

Thus, an allocation x is attainable for the coalition S if S can ensure its members $x(a)$, $a \in S$, by using its aggregate initial endowment .

An allocation $x \in X(\mathcal{E})$ is said to be *individually rational* if $e(a) \not\succ_a x(a)$ for a.a. $a \in A$. Thus, an allocation x is individually rational if there is no coalition with positive measure such that all agents in the coalition prefer their initial endowments to the bundle they obtain by x. An allocation $x \in X(\mathcal{E})$ is said to be *Pareto efficient* if there does not exist $y \in X(\mathcal{E})$ such that $y(a) \succ_a x(a)$ for a.a. $a \in A$. Thus an allocation x is Pareto efficient if it is impossible to distribute the total initial endowments in the economy such that almost all agents in A get bundles they prefer to the bundles obtained by x. When the consumers in an economy $\mathcal{E}$ have preferences in $\mathcal{P}^{*}_{mo}$, then we say that an allocation has *equal treatment* if $x(a) \sim_a x(b)$ for almost all $a, b \in A$ for which $(e(a), \succ_a) = (e(b), \succ_b)$.

2.1. Atomless Economies

An economy $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}^{\ell}_{+} \times \mathcal{P}_{mo}$ is called an *atomless economy* if $(A, \mathcal{A}, \lambda)$ is an atomless measure space. That is, for all $S \in \mathcal{A}$ with $\lambda(S) > 0$ there exists $B \subset S$, $B \in$

$\mathcal{A}$ such that $\lambda(B) > 0$ and $\lambda(S \setminus B) > 0$. Hence, an economy is atomless if any non-null coalition can be split into two non-null coalitions. Clearly, if an economy is atomless then each individual agent is a null set and there is necessarily a more than countable number of agents in the economy. Atomless economies were introduced by Aumann (1964) as a way to formalize that the economy consists of many (a continuum of) small agents. Modeling a real world economy as an atomless economy makes it endogenous that agents individually have no influence on the set of attainable allocations for any coalition. If an allocation is attainable for S and a null set of agents changes their consumption plan, the new allocation is again attainable for S.

A useful tool in analyzing atomless economies is Lyapunov's Theorem as introduced into economics by Vind (1964).

Theorem 1 **(Lyapunov)**

Consider a finite family of finite non-negative atomless measures $\mu = (\mu_1, \mu_2, ..., \mu_n)$ on the measurable space $(A, \mathcal{A})$. Then the range $R(\mu) = \{x \in \mathbb{R}^n \mid$ there exists $C \in \mathcal{A}$ where $x_h = \mu_h(C)$, $h = 1, ..., n\}$ is a compact and convex subset of $\mathbb{R}^n$.

Clearly, Lyapunov's Theorem implies that for an atomless economy $\mathcal{E}$ with consumers in $(A, \mathcal{A}, \lambda)$ and an integrable function $x : A \to \mathbb{R}^\ell$, $\{\int_S x d\lambda \mid S \in \mathcal{A}\}$ is a convex subset of $\mathbb{R}^\ell$. Moreover for any correspondence (set-valued function) $\phi : A \Rightarrow \mathbb{R}^\ell$, the set $\int_A \phi d\lambda = \{\int_A f d\lambda \mid f(a) \in \phi(a)$ a.a. $a \in A$ and f integrable$\}$ is convex.

2.2. Finite Economies

A finite economy is an economy $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}^\ell_+ \times \mathcal{P}_{mo}$ where A is a finite set, $\mathcal{A}$ is all subsets of A, and λ is the counting measure, that is, $\lambda S) = \frac{|S|}{|A|}$ for all $S \subset A$.

A useful tool in analyzing large finite economies is the Shapley-Folkman Theorem as introduced by Starr (1969).

Theorem 2 **(Shapley-Folkman)**

Let Z_i, $i = 1, ..., n$ be a family of non-empty subsets of $\mathbb{R}^\ell$ and let $u \in conv \sum_{i=1}^{n} Z_i$. Then there are points $u_i \in \text{conv} Z_i$, $i = 1, ..., n$, such that $x = \sum_{i=1}^{n} u_i$ with $u_i \in Z_i$ except for at most ℓ of the points.

Note in particular, that the number of exceptional points, that is, points which are not in Z_i, depends on the dimension ℓ of the Euclidean space but not on the number of sets in the family. The Shapley-Folkman Theorem is an approximate version of Lyapunov's Theorem. Consider for example the case where the sets Z_i, $i = 1, ..., n$, are uniformly bounded. Then the Euclidean distance between the convex hull of the sum of the sets and the sum itself is bounded independently of the number of sets in the family.

3. Walrasian Equilibrium

3.1. Walrasian Allocations

We shall now define the set of allocations, which can be obtained by the Walrasian institution. That is, attainable allocations that can be obtained by letting each consumer independently choose an optimal net trade in a *Walrasian market* $M(p) = \{z \in \mathbb{R}^\ell \mid p \cdot z \leq 0\}$.

Definition 2

Let $\mathcal{E}$ be an economy. The pair $(p, x) \in \mathbb{R}^\ell \setminus \{0\} \times X(\mathcal{E})$ consisting of a price system and an attainable allocation is a Walrasian Equilibrium for $\mathcal{E}$ if [(i)]$p(x(a) - e(a)) \leq 0$ for a.a. $a \in A$, $y \succ_a x(a) \Rightarrow p(y - e(a)) > 0$ a.a. $a \in A$.

A Walrasian allocation *is an allocation x for which there exists a price system p such that (p, x) is a Walrasian Equilibrium. We let* $W(\mathcal{E})$ *denote the set of Walrasian allocations for the economy* $\mathcal{E}$.

In a Walrasian equilibrium all consumers take the Walrasian market $M(p) = \{z \in \mathbb{R}^\ell \mid p \cdot z \leq 0\}$ with the price system p as given and choose net trades so as to maximize their preference relations. If the economy $\mathcal{E}$ is atomless then of course no agent will be able to manipulate the Walrasian price system. More precisely, assume that prices are set such that markets clear. Then the price system clears the markets independent of the action of an individual agent (and a null set of agents).

3.2. Strongly Fair Net Trades

An elementary characterization of a Walrasian allocation for a finite economy $\mathcal{E}$ is given in Schmeidler and Vind (1972).

Definition 3

Let $\mathcal{E}$ *be a finite economy. The allocation x has* strongly fair net trades *if for all agents* $a \in A$ *and all* $n_b \in \mathbf{Z}_+$

$$\sum_{b \in A} n_b(x(b) - e(b)) + e(a) \in \mathbb{R}^\ell_+ \Rightarrow \sum_{b \in A} n_b(x(b) - e(b)) + e(a) \not\succ_a x(a).$$

The idea behind the concept of strongly fair net trades is the following: Each agent a considers the net trades obtained by the agents in A, that is the set $Zx = \{x(b) - e(b) \in \mathbb{R}^\ell \mid b \in A\}$ of net trades revealed by x. If the institution leading to x is fair, then all the net trades in Z_x should be available to any of the consumers. Hence, in equilibrium, none of the consumers should prefer any of these net trades to the net trade they themselves have obtained. (This equilibrium condition leads to the concept of allocations having *fair net trades*.) However, one might argue that an agent should also be able to obtain a net trade which is the sum of net trades revealed by x, and also any net trade which is a linear combination of such net trades with non-negative integer weights. An agent just uses the market possibilities repeatedly. In equilibrium no agent should prefer such a combination of the net trades revealed by x. This is exactly what the condition in the definition of strongly fair net trades says.

Clearly, any Walrasian allocation has strongly fair net trades. Schmeidler and Vind (1972) show that apart from indivisibilities, this condition also characterizes a Walrasian allocation in the following sense. Assume that $X \subset \mathbb{R}^\ell$ is the marketed subset of the commodity space, that is, for any price system $p \in \mathbb{R}^\ell$ the Walrasian market given X equals $\{z \in X \mid p \cdot z \leq 0\}$. Thus for any price system p the consumers cannot choose net trades in the whole of $\mathbb{R}^\ell$ but only in the marketed space X. We can now define the set of Walrasian allocations relative to X. The definition of a Walrasian allocation above being the special case where $X = \mathbb{R}^\ell$. Vind and Schmeidler show that if the attainable allocation x has strongly fair net trades and reveals divisibility (for a precise definition see Schmeidler and Vind) then x is a Walrasian allocation relative to the smallest linear subspace of $\mathbb{R}^\ell$ containing $\{(x(a) - e(a)) \mid a \in A\} \cup \{c\}$ for any $c \in \mathbb{R}^\ell$, $c >> 0$. In particular, if the dimension of smallest linear subspace containing $\{(x(a) - e(a)) \mid a \in A\}$ has dimension $\ell - 1$, then x is a Walrasian allocation.

The main insight used in the proof of Schmeidler and Vind's theorem is that when x is an attainable allocation, then the set $\tilde{Z}_x = \{\sum_{b \in A} n_b(x(b) - e(b)) \mid n_b \in \mathbf{Z}_+\}$ with addition is a group. Clearly $\tilde{Z}_x$ is closed under addition and $0 \in \tilde{Z}_x$. To see that all $z \in \tilde{Z}_x$ have inverse elements in $\tilde{Z}_x$ consider any $z = \sum_{b \in A} n_b(x(b) - e(b)) \in \tilde{Z}_x$. Since $x(b) - e(b) = -\sum_{a \neq b}(x(a) - e(a))$ for all $b \in A$, then $-z$ is also in $\tilde{Z}_x$.

A theorem corresponding to Schmeidler and Vind's also holds for an atomless economy $\mathcal{E}$. Define for each attainable allocation x the net trade set $\bar{Z}_x = \{\int_S (x - e)\, d\lambda \mid S \in \mathcal{A}\}$. We now say that the allocation x has strongly fair net trades if, for no non-null coalition S, there exists an integrable function $y : S \to \mathbb{R}^\ell_+$ such that

[(i)] for all $S' \subset S$, $S' \in \mathcal{A}$, $\int_{S'} (y - e)\, d\lambda \in \bar{Z}_x$, and $y(a) \succ_a x(a)$ for a.a. $a \in S$.

It is easily seen that a Walrasian allocation has still strongly fair net trades. The opposite conclusion, namely that an attainable allocation x with strongly fair net trades is a Walrasian allocation relative to the smallest linear subspace of $\mathbb{R}^\ell$ containing $\bar{Z}_x \cup \{c\}$ for any $c \in \mathbb{R}^\ell$, $c >> 0$, also holds true. This follows, as in Schmeidler and Vind's theorem, since the set $\bar{Z}_x$ is compact and convex by Lyapunov's Theorem. Moreover, $\bar{Z}_x$ is symmetric since x is an attainable allocation, and clearly $0 \in \bar{Z}_x$.

In Vind (1978) the concept of a simple market and a corresponding equilibrium notion are defined. A market is simple if it contains any finite sum of elements of itself. Clearly, the set $\tilde{Z}_x$ defined above is an example of a simple market. Analogously with Schmeidler and Vind's Theorem, Vind obtains an equivalence result. The paper by McLennan and Sonnenschein (1991) also contains an equivalence result based on the structure of the set of net trades available to the agents. McLennan and Sonnenschein define a strategic market game with a continuum of agents and give conditions under which all subgame perfect equilibria of the market game yield Walrasian allocations.

4. Equivalences in Atomless Economies

4.1. The Core

The Core of an economy is defined analogously to the Core of a game. However, instead of considering utility profiles we look at the set of attainable allocations.

Definition 4

Let $\mathcal{E}$ be an economy and let x be an allocation for $\mathcal{E}$. We say that a non-null coalition $S \in \mathcal{A}$ can improve upon x, if there exists an attainable allocation $y : S \to \mathbb{R}_+^\ell$ for S with [(i)]y(a) $\succ$ x(a) for a.a. $a \in S$.

The Core of an economy $\mathcal{E}$ is defined as the set of attainable allocations for $\mathcal{E}$, which cannot be improved upon by any non-null coalition. Formally,

Definition 5

Let $\mathcal{E}$ be an economy. The allocation x is in the Core *if $x \in X(\mathcal{E})$ and if there does not exist a non-null coalition $S \in \mathcal{A}$, which can improve upon x. We let Core($\mathcal{E}$) denote the Core of the economy $\mathcal{E}$.*

The principles behind the Core are very different from the ones behind the Walrasian institution. An attainable allocation is in the Core if it is stable in the sense that there is no non-null coalition, which can redistribute its total initial endowments in such a way that almost every member of the coalition obtains a preferred commodity bundle. In the definition of the Core, no agent or coalition is restricted by a priori given market institutions. The only restriction on the attainable allocations for a coalition is the private property right to the initial endowments.

A Walrasian allocation is in the Core, but the very interesting result, shown by Aumann (1964), is that the set of Walrasian allocations and the Core coincide for atomless economies.

Theorem 3 **(Aumann's Core Equivalence Theorem)**

Let $\mathcal{E} : (A, \mathcal{A}, \lambda) \to \mathbb{R}_+^\ell \times \mathcal{P}_{mo}$ be an atomless economy. Then $W(\mathcal{E}) = Core(\mathcal{E})$.

The version we have given of Aumann's equivalence theorem is due to Schmeidler (1969), who realized that Aumann's theorem also holds true if agents have non-complete preferences.

The proof of Aumann's theorem is quite easy when Lyapunov's theorem is used. Consider an allocation $x \in Core(\mathcal{E})$. For each coalition S, define the set of preferred net trades by $\Psi_S = \{z \in \mathbb{R}^\ell \mid \exists y : S \to \mathbb{R}^\ell$ such that $y(a) + e(a) \succ_a x(a)$ a.a. $a \in S$ and $\int_S y\, d\lambda = z\}$. Let $\Psi = \bigcup_{\lambda(S)>0} \Psi_S$. Clearly, $0 \notin \Psi$ since $x \in Core(\mathcal{E})$. Moreover Ψ is convex. Indeed define the modified preferred correspondence $\psi : A \Rightarrow \mathbb{R}^\ell$ by $\psi(a) = \{z \in \mathbb{R}^\ell \mid z + e(a) \succ_a x(a)\} \cup \{0\}$. Then $\Psi = \int_A \psi\, d\lambda$ and $\int_A \psi\, d\lambda$ is convex by Lyapunov's Theorem. By monotonicity $0 \in bd\Psi$ and also $\Psi \cap \{u \in \mathbb{R}^\ell \mid u << 0\} = \emptyset$. Hence, there exists a price system $p \in \mathbb{R}^\ell \setminus \{0\}$ such that $\{0\}$ is separated from Ψ. At the price system p any net trade, which is preferred by a non-null coalition has a non-negative value. Monotonicity implies that $p \in \mathbb{R}^\ell_+$. The fact that x is attainable allows us to show that (p, x) is a Walrasian equilibrium.

In general, the Core equivalence theorem does not hold when the measure space of agents has atoms, for example if the economy has a finite number of consumers. However there have are several papers (for example Gabszewicz and Mertens (1971) and Shitovitz (1973)) in which the authors obtain Core equivalence assuming that for each atom there exists agents in the atomless part with the same characteristics, and that the atoms are not too big relative to the atomless coalitions of similar consumers.

The Core equivalence theorem has been extended in many different directions. For example it was shown by Schmeidler (1972) that an attainable allocation which is not in the Core can be improved upon by an arbitrarily small coalition, and by Grodal (1972) that the small coalition can further be restricted to consist of small groups of similar agents. Moreover, Vind (1972) shows that if an attainable allocation can be improved upon then it can also be improved upon by an arbitrarily large coalition.

Recently, Ellickson et al. (1999) showed that the Core equivalence theorem extends to club economies. They assume that consumers trade private commodities on a market, can belong to several clubs, and care about the characteristics of the other members of their clubs. In an attainable allocation, all seats in club type are filled or no seats are filled. In a *Club* (*Walrasian*) *Equilibrium*, not only the private commodities are priced but also membership of the clubs. The Core is defined analogously to the definition of the Core in a private goods economy. An attainable allocation is in the Core if no non-null coalition can form clubs and distribute private commodity bundles by using the coalition's initial endowments and thereby ensure that almost all of its members get combinations of private commodities and club memberships which they prefer.

Aumann's (1964) theorem has also been extended to infinite dimensional commodity spaces. For example, Bewley (1973) extended Aumann's Core Equivalence Theorem to the case where the commodity space is the set of essential bounded measurable functions on a measure space, L_∞. However, as has been shown by Ostroy and Zame

(1994), the Core equivalence in an economy with a continuum of agents and a continuum of commodities is much more subtle than the equivalence when the number of commodities is finite. When there is a continuum of commodities, an atomless measure space of agents does not, in general, imply Core equivalence.

However, in economies with finitely many commodities, the Core equivalence result for economies with many small agents is a very robust. Almost no assumptions on the preferences are needed, in particular, preferences are not assumed to be convex or complete and the equivalence holds with many types of restrictions on the coalitions which are "allowed" to improve upon an attainable allocation. Moreover, the equivalence holds when there are limited externalities among consumers, as in the case of club economies. (A preference relation $\succ$ is convex if $\{y \in \mathbb{R}_+^\ell \mid y \succ x\}$ is convex for all $x \in \mathbb{R}_+^\ell$.)

4.2. The Bargaining Set

The Core as defined above is based on a veto power from any coalition; as soon as a coalition can improve upon an attainable allocation, this allocation is dismissed. However, one might argue that a coalition, which can improve upon an allocation only has an objection against the allocation and that this objection might be met with a counter-objection. This argument leads to the Bargaining Set. The *Bargaining Set* for cooperative games was introduced by Aumann and Maschler (1964). For an atomless economy the Bargaining Set was first defined by MasColell (1989) and the definition is repeated here.

Definition 6

Let $\mathcal{E}$ be an economy in which the consumers have preferences in $\mathcal{P}_{mo}^*$. The pair (S, y), where $S \in \mathcal{A}$ and $y : S \to \mathbb{R}_+^\ell$, is integrable, is an objection *to the attainable allocation x if $\int_S y\, d\lambda \leq \int_S e\, d\lambda$, and $y(a) \succsim_a x(a)$ for a.a. $a \in S$ and $\lambda(\{a \in S \mid y(a) \succ_a x(a)\}) > 0$.*

Definition 7

Let (S, y) be an objection to the attainable allocation x. The pair (T, z), where $T \in \mathcal{A}$ and $z : T \to \mathbb{R}_+^\ell$ is integrable, is a counter-objection to (S, y) if $\int_T z\, d\lambda \leq \int_T e\, d\lambda$, $\lambda(T) > 0$, and [(i)]$z(a) \succ_a y(a)$ for a.a. $a \in T \cap S$ $z(a) \succ_a x(a)$ for a.a. $a \in T \setminus S$.

Definition 8

An objection (S, y) is said to be justified if there is no counter-objection to it. The Mas-Colell Bargaining Set is the set of attainable allocations against which there is no justified objection.

Clearly, the Mas-Colell Bargaining Set contains the Core, and hence the set of Walrasian allocations. The main result in Mas-Colell (1989) is that in an atomless economy the converse is also true. Thus, even if only justified objections are allowed, we again get an equivalence result.

Theorem 4 **(Mas-Colell's Bargaining Set Equivalence Theorem)**

Let $\mathcal{E}$ be an atomless economy and assume that consumers have preferences in $\mathcal{P}^*_{mo}$. Then an allocation x is in the Mas-Colell Bargaining Set if and only if it is a Walrasian allocation.

The proof of the above equivalence is based on an interesting observation, namely that there are only few justified objections and that these can be characterized by a price system. Indeed, define

Definition 9

The objection (S, y) to the attainable allocation x is a Walrasian objection *if there is a price system $p \neq 0$ such that*

[(*i*)]$v \succsim_a y(a) \Rightarrow p \cdot v \geq p \cdot e(a)$ *for a.a.* $a \in S$, $v \succsim_a x(a) \Rightarrow p \cdot v \geq p \cdot e(a)$ *for a.a.* $a \in A \setminus S$.

The connection between justified objections and Walrasian objections are given by the following proposition

Proposition

Let $\mathcal{E}$ be an atomless economy and assume that consumers have preferences in $\mathcal{P}^*_{mo}$. An objection (S, y) is justified if and only if it is Walrasian.

Thus, coalitions *S* which not only can improve upon a given allocation *x* but which are part of justified objections (*S, y*) to *x* are much more determinate. Given an attainable allocation *x* and a price system *p* almost all agents $a \in A$ who prefer their Walrasian demands at *p* to *x*(*a*) must be part of the group of agents attempting to make an objection to *x* sustainable by the price system *p*.

The proposition above also shows why stronger conditions are needed to obtain the Bargaining Set equivalence than the Core equivalence. For any attainable allocation *x* let $v_x: A \times \Delta \rightarrow \mathbb{R}$ be defined by $v_x(a, p) = \inf\{p \cdot z \mid z + e(a) \succ_a x(a)\}$. Moreover let $C_x(p) = \{a \in A \mid v_x(a, p) < 0\}$ and $D_x(p) = \{a \in A \mid v_x(a, p) \leq 0\}$. Clearly, the attainable allocation *x* is not a Walrasian allocation if and only if for all $p \in \Delta$, $\lambda(C_x(p)) > 0$. Hence, in order to get the Bargaining Set equivalence it is necessary that for any attainable allocation which is not a Walrasian allocation, there exists $\overline{p} \in \Delta$ and $S \in \mathcal{A}$ such that $C_x(\overline{p}) \subset S \subset D_x(\overline{p})$an such that the following two conditions are satisfied

[(*i*)]*For all* $a \in S \setminus C_x(\overline{p})$ *there exists a maximal element* $y(a)$ *for* $\succ_a$ *in the budget set*

$B_a(\overline{p}) = \{u \in \mathbb{R}^{\ell}_{+} \mid \overline{p}u \leq \overline{p}e(a)\}$. For all $a \in C_x(\overline{p})$ there exists a maximal element $y(a)$ for $\succ_a$ in the budget set $B_a(\overline{p})$ such that $y(a)$ $x(a)$ and, moreover, there must exist $F \in \mathcal{A}$, $\lambda(F) > 0$, with $F \subset \{a \in C_x(\overline{p}) \mid y(a) \succ_a x(a)\}$.

Hence, in order to obtain the Bargaining Set equivalence we need that there exist maximal elements in the consumers' budget sets. However, it is well known that this is only guaranteed if either $\succ_a$ is transitive or convex. Grodal (1988) gives equivalence theorems for economies in which the consumers have non-complete preference relations along these lines.

Schjødt and Sloth (1994) analyze the Bargaining Set when there are restrictions on the size of the coalitions which can make objections and also, on the coalitions which can make counter-objections. They show, not surprisingly, that if one restricts the measure of the coalitions which can enter into objections, as well as the measure of the coalitions which can enter into counter-objections, then the modified Bargaining Set is strictly larger than the Mas-Colell Bargaining Set. Hence, in contrast to the Core, this modified Bargaining Set will not entail equivalence to the set of Walrasian allocations.

The original Aumann and Maschler (1964) Bargaining Set requires that there is a player who is the leader of an objection (S, y) and that a counter-objection against (S, y) does not contain the leader. Geanakoplos (1978) defines a modification of the Aumann and Maschler Bargaining Set for a finite exchange economy with transferable utility, still requiring that there is "a leader" of an objection.

However, in the Geanakoplos Bargaining Set "the leader" is a fixed fraction of the agents. Hence, as the number of players in the economy increases, the number of agents in the group of leaders is also allowed to grow.

Let $\delta > 0$ and define a δ-objection to the attainable allocation x as a triple (U, S, y) such that $U \subset S$, $\lambda(U) \leq \delta$, and (S, y) is an objection to x. A counter-objection to (U, S, y) is a counter-objection (T, z) to (S, y) such that $U \cap T = \emptyset$. A δ-objection is justified if there is no counter-objection to it. Now define the Geanakoplos δ-Bargaining Set as all attainable allocations x for which every δ-objection has a counter-objection. Clearly, the Mas-Colell bargaining Set contains the Geanakoplos δ-Bargaining Set for all $\delta > 0$, and thus the equivalence result again holds true for the Geanakoplos δ-Bargaining Set.

It is easily seen from the proof of Mas-Colell's Bargaining Set equivalence theorem that the equivalence is not true if the requirement to an objection (S, y) to x is strengthened to $y(a) \succ_a x(a)$ for a.a. $a \in S$. However, as noted by Mas-Colell (1989), if one uses this strengthened request to a δ-objection (U, S, y) to x, then we obtain the following equivalence: Given any $\delta > 0$, if there is no strengthened justified δ-objection against the attainable allocation x, then x is a Walrasian allocation.

4.3. The Value

In this section we consider another fundamental equivalence, namely the equivalence between the set of Value allocations, as defined in Aumann (1975), and the set of Walrasian allocations. The Value allocations originate from Shapley's (1953) definition of the Value of a transferable utility (TU) game. The Shapley Value is based on agents' marginal worth to the coalitions in which they are members. Thus, the solution concept is quite different from the Core and the Bargaining Set.

Before defining the Value allocations for an atomless economy we first recall Shapley's definition of the Value of a finite TU game. Let N be a finite set of players and let $\mathcal{N} = 2^N$ be the coalitions. A *game* v on N is a function $v : \mathcal{N} \to \mathbb{R}$, with $v(\emptyset) = 0$. Let V be the set of games on N. For all $S \in \mathcal{N}$ we call $v(S)$ the worth of S. A *payoff* is any measure on $\mathcal{N}$. Let $\mathcal{M}$ be the set of measures on $\mathcal{N}$. A *null player* in the game v is a player $a \in A$ with $v(S) = v(S \cup \{i\})$ for all $S \in \mathcal{N}$. Players a and b are called *substitutes* if $v(S \cup \{a\}) = v(S \cup \{b\})$ for all $S \in \mathcal{N}$ with $a, b \notin S$. The *Value* φ is a function $\varphi : V \to \mathcal{M}$, $v \mapsto \varphi v$ which satisfies the following conditions:

[(*i*)]*Additivity:* $\varphi(v + w) = \varphi(v + w)$, *Symmetry:* $(\varphi v)(\{a\}) = (\varphi v)(\{b\})$ *whenever a and b are substitutes, Efficiency:* $(\varphi v)(N) = v(N)$ *Null player condition:* $(\varphi v)(\{a\}) = 0$ *whenever a is a null player in v.*

Shapley (1953) showed that there is one and only one Value φ. It is given by the formula $(\phi v)(\{a\}) = E(v(S_a \cup \{a\}) - v(S_a))$, where S_a is the set of all players preceding a in a random order on N, and E is the expectation operator when all $|N|!$ such orders are assigned equal probability. Hence, the Shapley Value assigns to each player his marginal contribution averaged over all random orders on the set of agents.

As we are going to consider atomless economies, we now recall Kannai's (1966) definition of the asymptotic Value of a continuous game. In order to define the Value of a continuum transferable utility game, one simply approximates the continuum game with finite games and uses a limit argument to obtain a Value of the infinite game. Since we shall not use the explicit construction, we refer the reader to Kannai (1966) and Aumann and Shapley (1971) for the exact construction. This Value ϕv of a game v, constructed by taking the limits of finite approximations to v, is called the *asymptotic Value* of v and it is a finitely additive measure on $(A, \mathcal{A})$.

In order to define the Value allocations of our atomless economy we make the following definition.

Definition 10

A family of utility functions $(u_a)_{a \in A}$, *where* u_a: $\mathbb{R}^{\ell}_{+} \to \mathbb{R}$ *is called* bounded differentiable *if the following conditions are satisfied:*

[(*i*)]u_a *is* C^1 *for all* $a \in A$. *The function* $u_{\cdot}(\cdot) : A \times \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ *is jointly measurable. The family of functions* $(u_a)_{a \in A}$ *are uniformly bounded. The derivatives* u'_a *are, in compact sets, uniformly bounded and are uniformly bounded away from* 0.

For each family $U = (u_a)_{a \in A}$ of bounded differentiable representation of the preferences $(\succ_a)_{a \in A}$ we define a game v_U on $(A, \mathcal{A})$ by letting

$$v_U(S) = \max\left\{ \int_S u_a(x(a))d\lambda \mid \int_S x(a)d\lambda = \int_S e(a)d\lambda, x(a) \in \mathbb{R}_+^\ell \text{ a.a. } a \in S \right\}$$

The number $v_U(S)$ can be interpreted as the maximum welfare, measured in the family of utility representations U, the coalition S can guarantee itself no matter what the other players do.

Clearly, the game v_U depends on the family U of representations of the agents' preference relations. We now define the set of Value allocations by considering different bounded differentiable representations of the consumers' preferences. Our assumptions on U imply that the asymptotic Values φv_U of the games v_U are well defined.

Definition 11

An attainable allocation x for the economy $\mathcal{E}$ is a Value allocation *if there exists a bounded differentiable family U of utility representation for the consumers' preferences such that*

$$(\varphi v_U)(S) = \int_S u_a(x(a))d\lambda \text{ for all } S \in \mathcal{A}\,..$$

Thus, a Value allocation is an attainable allocation for the economy $\mathcal{E}$ having the property that there exists utility representations for the consumers' preferences such that when these utility functions are used to define aggregate welfare, almost all agents get exactly a utility level equal to their own marginal contribution to the welfare of the society. Note, that even if the game v_U is defined by making transfers of utility among agents, no transfers take place in a Value allocation.

Aumann (1975) introduces the assumption that preferences of the agents are uniformly smooth. The family $(\succ_a)_{a \in A}$ is said to be *uniformly smooth* if each of the preferences are smooth and there exists a bounded differentiable family $(u_a)_{a \in A}$ of utility functions representing consumers' preferences such that the family of Gaussian curvatures $(u_a^c)_{a \in A}$ are uniformly bounded away from 0 in compact sets, and the family of second derivatives $(u_a'')_{a \in A}$ are uniformly bounded in compact sets.

We now have the following theorem.

Theorem 5 **(Aumann's Value Equivalence Theorem)**

Let $\mathcal{E}$ be an atomless economy in which the measurable space $(A, \mathcal{A})$ is isomorphic to [0, 1] *with the Borel σ-field. Assume that consumers have uniformly smooth preferences* *and that e is uniformly bounded. Then the set of Value allocations coincides with the set of Walrasian allocations.*

The proof of Aumann's Value equivalence theorem is based on the theory of non-atomic games developed in Aumann and Shapley (1971). Thus, the proof is very different from the proof of the Core and Bargaining Set equivalence theorems. Moreover, the equivalence rests on much stronger assumptions on the atomless economy than the Core and Bargaining Set equivalences. However, Aumann (1975) gives counter examples showing that the Value equivalence does not hold without differentiability assumptions. Also, it should be noted that the Walrasian allocations are not, in general, a subset of the set of Value Allocations.

4.4. Axiomatic Approach to the Equivalence Phenomenon

Dubey and Neyman (1997) have given an axiomatic foundation for the Core equivalence theorem and the Value equivalence theorem. They consider a family of economies $\tilde{\mathcal{M}}$ with a fixed atomless measure space $(A, \mathcal{A}, \lambda)$ of agents and impose restrictions on how the set of "equilibrium allocations" behave when different mappings $\mathcal{E}$ are considered. Clearly, this approach is very different from considering one economy and comparing different institutions in this one economy.

As in Aumann's Value equivalence theorem, it is assumed that the measurable space $(A, \mathcal{A})$ of agents is isomorphic to [0, 1] with the Borel σ-field. All economies in $\tilde{\mathcal{M}}$ have a finite dimensional commodity space. Thus, the number of commodities is allowed to vary. Moreover, consumers in an economy in $\tilde{\mathcal{M}}$ are assumed to have uniformly bounded initial endowments and to have preferences in $\mathcal{P}^*_{mo}$ which are smooth. (The conditions in Dubey and Neyman (1997) on the preferences are slightly weaker.)

Let F be the set of allocations for economies in $\tilde{\mathcal{M}}$. For each $E \in \tilde{\mathcal{M}}$ define $F(E) \subset F$ as the set of attainable allocations for $\mathcal{E}$ that are bounded, individually rational, and Pareto efficient. Dubey and Neyman (1997) define admissible correspondences in the following way:

Definition 12.

The correspondence $\tilde{\varphi}: \tilde{\mathcal{M}} \to F$ *is* admissible *if the values of* $\tilde{\varphi}$ *are non-empty,* $\tilde{\varphi}(\mathcal{E}) \subset F(\mathcal{E})$ *and* $x \in \tilde{\varphi}(\mathcal{E})$, $y \in F(\mathcal{E})$, *and* $x(a) \sim_a y(a)$ *for a.a.* $a \in A$ *imply* $y \in \tilde{\varphi}(\mathcal{E})$.

Dubey and Neyman introduce the following four axioms:

[(*i*)]*Anonymity: Only the characteristics of the agents matter. Equity: For any economy there exists an allocation in* $\tilde{\varphi}(\mathcal{E})$ *such that almost all agents with identical characteristics get equivalent bundles. Consistency: Consider three economies,* $\mathcal{E}^i$, i = 1, 2, 3, *with respectively l, k and l + k commodities. Let the initial endowments in* $\mathcal{E}^i$ *be* e^i *and assume* $e^3 = (e^1, e^2) \in \mathbb{R}^\ell_+ \times \mathbb{R}^k_+$. *Moreover, assume that the preferences of the* *agents $\mathcal{E}^3$ satisfy the independence axiom with respect to the set of commodities in $\mathcal{E}^2$ and $\mathcal{E}^1$ respectively, and that the preferences in $\mathcal{E}^1$ and $\mathcal{E}^2$ are consistent with the preferences in $\mathcal{E}^3$. The consistence axiom states:*

If $x \in \tilde{\varphi}(\mathcal{E}^1)$, $y \in \tilde{\varphi}(\mathcal{E}^2)$ and (x, y) is Pareto efficient in $\mathcal{E}^3$, then $(x, y) \in \tilde{\varphi}(\mathcal{E}^3)$. Moreover, if $(x, e^2) \in \tilde{\varphi}(\mathcal{E}^3)$ then $x \in \tilde{\varphi}(\mathcal{E}^1)$.

Restricted Continuity: Detailing this axiom goes beyond the scope of this chapter. Briefly, the idea is to introduce the concept of "TU-like at a positive level" and to demand that the correspondence $\tilde{\varphi}$ induces a correspondence with values in the payoff space that are continuous on the set of such economies.

Theorem 6 **(Dubey and Neyman)**

There is one and only one admissible correspondence $\tilde{\varphi}$ defined on $\tilde{\mathcal{M}}$ which satisfies Anonymity, Equity, Consistency, and Restricted Continuity, and that is the correspondence which maps each of the economies into the set of Walrasian allocations for $\mathcal{E}$.

It is easily shown that the Core and the Value correspondences also satisfy the four axioms. Hence, Dubey and Neyman have shown that the equivalence principle between the three concepts holds for the class of atomless economies $\tilde{\mathcal{M}}$. Clearly, this space of economies is much smaller than the space where the Core equivalence holds; but it is comparable with the space of economies used in Aumann's Value equivalence theorem.

5. Approximations to Equivalence: Large Finite Economies

In this section we consider large finite economies.

5.1. The Core

Clearly, to describe a real world economy as an atomless economy is an abstraction. Hence, conclusions derived from atomless economies ought to be followed up by an investigation into whether the conclusions are true, in some approximate version, for large finite economies. There is an extensive literature on the connection between the set of Walrasian allocations or Walras-like allocations and the Core in large, but finite economies. The origin of the work can be traced back to Edgeworth (1881) (Vind (1995) argues that Edgeworth did not analyze the Core but the set of Exchange Equilibria.). For an extensive overview of the literature see Anderson (1992). The most influential paper is Debreu and Scarf (1963). In that paper Debreu and Scarf consider a sequence of economies, constructed by letting be the *n*-fold replica of a finite economy in which consumers' preferences are represented by strictly quasiconcave utility functions. Debreu and Scarf prove that *Core*() is decreasing and that *Core*() = *W*(). Hence the distance between the set of Core allocations and the Walrasian allocations converge to 0 along the sequence.

Since Debreu and Scarf's paper many different versions of their convergence result have been obtained. The assumptions have be weakened and one has obtained convergence for non-replica sequences of economies with a growing number of agents. The reader is referred to the classical book Hildenbrand (1974) and to the survey by Anderson (1992). A natural question to ask in relation to Debreu and Scarf's theorem is how fast the sequence $Core(\mathcal{E}^n)$ converges to $W(\mathcal{E}^1)$. Debreu (1975) shows that for replica of a smooth and regular economy, the speed of convergence is $o(n)$; a theorem which is generalized to arbitrary sequences in Grodal (1975).

In general there are many ways one can define an approximation to the Core equivalence theorem for large economies. In Debreu and Scarf's theorem we have a very strong kind of approximate Core equivalence when the replication is sufficiently large. The distance between the set of Walrasian allocations and the Core converges to zero. As shown by Bewley (1973) such a strong approximate Core equivalence result cannot, in general, be obtained for large finite economies.

We define approximate equivalence if there exists for any Core allocation, a price system such that the Core allocation can be approximately decentralized by this price system. Thus we shall state the classical theorem from Anderson (1978), which gives a very elegant result on approximate decentralization of Core allocations in a finite economy in which consumers have preferences in $\mathcal{P}_{mo}$. (For another and independent version of the theorem see Dierker (1975).)

Theorem 7 **(Anderson)**

Let $\mathcal{E} : A \in \mathbb{R}^{\ell}_{+} \times \mathcal{P}_{mo}$ *be a finite economy and let* $x \in Core(\mathcal{E})$. *Then there exists* $p \in \Delta$ *such that*

$[(i)]\ \sum_{a\in A}|p \cdot (x(a) - e(a))| \leq 2\ell \max\{ \|e(a)\| \mid a \in A\} \sum_{a\in A}|\inf\{p \cdot (y - e(a))| \mid y \succ_a x(a)\}| \leq 2\ell \max\{ \|e(a)\| \mid a \in A\}$.

The proof of the theorem follows Schmeidler's (1969) proof of Aumann's Core equivalence theorem. However, instead of using Lyapunov's Theorem one uses the Shapley-Folkman Theorem. Let $M = \ell \max\{\|e(a)\| \mid a \in A\}$ and consider an allocation $x \in Core(\mathcal{E})$. Define the modified preferred correspondence $\psi : A \Rightarrow \mathbb{R}^{\ell}$ by $\psi(a) = \{z \in \mathbb{R}^{\ell} \mid z + e(a) \succ_a \} \cup \{0\}$. Let $\Psi = \sum_{a\in A} \psi(a)$ and let $\Omega_{-} = \{z \in \mathbb{R}^{\ell} \mid x << M(1,\cdots,1)\}$. The main step in the proof is to show that conv $\Psi \cap \Omega_{-} = \emptyset$. Assume to the contrary that $z \in$ conv $\Psi\ n\ \Omega_{-}$. As $z \in$ conv Ψ, Shapley-Folkman's Theorem states that there exists $(z(a))_{a\in A}$ with $z(a) \in$ conv $\psi(a)$ such that $\sum_{a\in A} z(a) = z$ and $z(a) \in \psi(a)$ for all but ℓ consumers. Let $\{a_1, \ldots, a_k\}$, $k \leq \ell$, be the exceptional agents where $z(a) \notin \psi(a)$. Now consider the coalition $B = \{a \in A | a \notin \{a_1, \ldots, a_k\}$ and $z(a) \neq 0\}$. For all agents $a \in B$ we have $z(a) + e(a) \succ_a x(a)$. Moreover as $z \in \Omega_{-}$ and $z(a) \geq -\max\{\|e(a)\| \mid a \in A\}(1, \ldots, 1)$ for all $a \in A$ then $\sum_{a\in B} z(a) << 0$. Thus $B \neq \emptyset$ and the coalition B can improve upon the allocation x. Hence we have a contradiction. As in the proof of Aumann's Core equivalence theorem we now obtain the existence of a price system $p \neq 0$ that separates Ω_- from the aggregate preferred set of net trades, Ψ. By monotonicity we can choose $p \in \Delta$. Again, the conclusions in the theorem follow from the separating property of p.

It should be noted that the right hand side of the inequalities in statements (i) and (ii) only depend on the dimension of the commodity space and the maximum size of agents' endowments. Define a measure ρ for how well a price system p decentralizes an allocation $x : A \to \mathbb{R}^\ell$ by

$$\rho(p,x) = \frac{1}{|A|}\left(\sum_{a\in A} |\, p\cdot(x(a)-e(a))\,| + \sum_{a\in A} |\, \inf\{p\cdot(y-e(a)) \mid y \succ_a x(a)\}\,|\right).$$

Anderson's Theorem then implies, that for any allocation x in the Core($\mathcal{E}$), there exists a price system $p \in \Delta$ such that $\rho(x, p) \leq \frac{1}{|A|} 4\ell \max\{\| e(a) \| \mid a \in A\}$. That is, we can find a price system such that, on average, the bundles $x(a)$, $a \in A$ are demand-like.

In Ellickson et al. (1999) it is shown that Anderson's Core decentralization result extends to Club economies. In Club economies the proof is, however, more complicated. Indeed, since the consumers who have $z(a) \in \text{conv } \psi(a) \setminus \psi(a) \cup \{0\}$ also consume club memberships, one cannot just dismiss these consumers when forming a coalition B that can improve upon x.

Anderson and Zame (1997) extend Anderson's approximate decentralization theorem to the case where the commodity space is the set of integrable functions on a finite measure space. However, they also give several examples of the failure of Core convergence. Hence, when the commodity space is infinitely dimensional, the Core convergence also becomes more subtle.

5.2. The Bargaining Set

Since we have equivalence between the Mas-Colell Bargaining Set, the Geanakoplos δ-Bargaining Set, and the set of Walrasian allocations in an atomless economy, it is natural to expect that approximate decentralization by prices can be obtained for large finite economies.

In a recent paper, however, Anderson, Trockel, and Zhou (1997) show that this is not the case for the Mas-Colell Bargaining Set. They give an example of a replica sequence of economies $\mathcal{E}^n$ for which there is a unique Walrasian equilibrium for all n. However, the measure of the set of individual rational, Pareto efficient, and equal treatment allocations, which are not in the Mas-Colell Bargaining Set, goes to zero as the economy is replicated. Hence the example shows very forcefully the non-convergence of the Mas-Colell Bargaining Set.

In the example, all agents have the same Cobb-Douglas utility function and the economy $\mathcal{E}^1$, which is replicated is regular. Indeed the basic economy $\mathcal{E}^1$ is the following: There are two commodities and two agents, $A = \{1, 2\}$. Both agents have the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ with $u(x, y) = \sqrt{xy}$. The initial endowments are respectively $e_1 = (3, 1)$ and $e_2 = (1, 3)$. In the economy $\mathcal{E}^1$ there is a unique Walrasian allocation $x(1) = x(2) = (2, 2)$.

Let $\mathcal{E}^n$ be the n-fold replica of the economy $\mathcal{E}^1$ and define for each $\xi \in [\sqrt{3}, 4 - \sqrt{3}]$ the attainable allocation x^n_ξ in $\mathcal{E}^n$ in which all consumers of type 1 get (ξ, ξ) and all consumers of type 2 get $(4 - \xi, 4 - \xi)$. The allocations x^n_ξ are individually rational, Pareto efficient, and have equal treatment. Let λ denote the Lebesgue measure on $\mathbb{R}$ and let $\mathcal{B}^n$ denote the Mas-Colell Bargaining Set of $\mathcal{E}^n$.

Theorem 8 **(Anderson, Trockel, and Zhou)**

For the replica sequence corresponding to $\mathcal{E}^{1\,1}$ described above the following hold:

[(*i*)]*For all* $\xi \in [\sqrt{3}, 4 - \sqrt{3}]$:
$|\{n \in \mathbf{Z}_+ \mid x^n_\xi \in \mathcal{B}^n\}| = \infty$ *There is a constant C such that for all* $n \in \mathbf{Z}_+$:
$\lambda(\{\xi \in [\sqrt{3}, 4 - \sqrt{3}]) \mid x^n_\xi \notin \mathcal{B}^n\}) \leq \frac{C}{\sqrt{n}}$

Thus, in the example, the set of individually rational, Pareto efficient equaltreatment allocations which are in the Mas-Colell Bargaining set converges in the Hausdorff distance to the set of all individually rational, Pareto efficient equaltreatment allocations. Nevertheless there is a unique Walrasian allocation.

The non-convergence in the example is driven by an integer problem. Hence, the authors leave open the possibility that sequences of non-replica economies might have better behaved Mas-Colell Bargaining Sets. Clearly, replica economies have the very special property that the agents' characteristics consist of a finite number of points and this set does not change along the sequence.

In Anderson (1998) it is shown that the Geanakoplos δ-Bargaining Set, and also the Aumann and Maschler bargaining Set, have qualitatively better convergence properties than the Mas-Colell Bargaining Set. Thus Anderson's result shows that it makes a fundamental difference for the convergence of Bargaining sets whether, as in the Aumann and Maschler and the Geanakoplos δ-Bargaining sets, one requires that there is a group of leaders or not. Here we just state one of the convergence results, namely one of Anderson's convergence results for the Geanakoplos δ-Bargaining Set. For any $\delta > 0$, let $\mathcal{B}_\delta(\mathcal{E})$ denote the Geanakoplos δ-Bargaining set of the economy $\mathcal{E}$.

Theorem 9 **(Anderson)**

Consider a sequence of finite economies $\mathcal{E}^n$, *where* $\mathcal{E}^n : A^n \to \mathcal{P}^*_{mo} \times \mathbb{R}^\ell_+$ *and*

$a \mapsto (\succ_a^n, e^n(a))$, *that satisfies*:

$[(i)] |A^n| \to \infty \frac{|S^n|}{|A^n|} \to 0 \Rightarrow \frac{1}{|A^n|} \sum_{a \in S^n} e^n(a) \to 0$, *and for all* κ *there exists a compact set* $K \subset \mathcal{P}_{mo}^*$ *such that*

$$\frac{|\{a \in A^n \mid \succ_a \in K\}|}{A^n} > 1 - \kappa .$$

($\mathcal{P}_{mo}^*$ is endowed with the topology of closed convergence. (See Hildenbrand 1974)) *Then there exists a sequence* $\delta^n \to 0$ *such that for every sequence* $\hat{\delta}^n$ *with* $\hat{\delta}^n \geq \delta^n$ *and every sequence* $(x^n)_n$ *with* $x^n \in \mathcal{B}_{\hat{\delta}^n}(\mathcal{E}^n)$, *there exists a sequence of prices* $(p^n)_n$ *with* $p^n \in \Delta$ *such that* $\rho(p^n, x^n) \to 0$.

Thus, Anderson has shown that if the sequence $(\hat{\delta}^n)_n$ does not converge too quickly towards 0 then, when the economy is sufficiently large, elements in the Geanakoplos $\hat{\delta}^n$-Bargaining Set can approximately be decentralized with prices. Again, this of course does not imply that the Geanakoplos $\hat{\delta}^n$-Bargaining Set and the set of Walrasian allocations are close to each other for large finite economies.

6. Strategic Behavior and Walrasian Equilibria

We now ask whether one can also characterize the set of Walrasian equilibria of an economy, as the set of Nash equilibria in a suitably defined non-cooperative game or non-cooperative generalized game. A non-cooperative generalized game is a non-cooperative game in which the feasible strategies of a player depend on the players' strategy profile. For each player, $a \in A$ there is a strategy set S_a and a constraint correspondence $\beta_a : \prod_{a \in A} S_a \Rightarrow S_a$, which maps a strategy profile $s = (s_a)_{a \in A}$ into the set of strategies $\beta_a(s) \subset S_a$, that is feasible for a given the profile s. Moreover, there is an outcome function $f : \prod_{a \in A} S_a \to \prod_{a \in A} \mathbb{R}^\ell$. A *Nash equilibrium* is a strategy profile $s = (s_a)_{a \in A} \in \prod_{a \in A} \beta_a(s)$ such that there does not exist for any player $s_a' \in \beta_a(s)$ with $f_a(s_{-a}, s_a') \succ_a f_a(s)$. (We let s_{-a} denote the profile of strategies for all players $b \neq a$.)

It is well known from Arrow and Debreu (1954) that, for a finite economy $\mathcal{E}$, one can construct a generalized game such that the Nash equilibria in the game equals the set of Walrasian equilibria of $\mathcal{E}$. The generalized game corresponding to the economy $\mathcal{E}$ is constructed by adding a player- the price maker. The strategy set of this additional player is the positive price simplex Δ and the additional player's preference relation on the price simplex is such that the player maximizes the value of the excess demand of the original consumers.

However, still using generalized games, one can also obtain the set of Walrasian Equilibria as the set of Nash outcomes without adding a player. For simplicity we assume that the players' consumption sets have been extended to $\mathbb{R}^\ell$. Assume that each agent chooses a Walrasian Market that is a normalized price system, and also a net trade in each of the other agents' markets. Hence, the strategy set of each agent is $S_a = \Delta \times \prod_{b\in A}\mathbb{R}^\ell$. Given the strategy profile$(p_a, (z_{ab})_{b\in A})_{a\in A} \in \prod_{a\in A} S_a$, the value of agent a's constraint correspondence is given by

$$\beta_a((p_a,(z_{ab})_{b\in A}) = \Delta \times \{(z_{ab})_{b\in A} \mid p_b \cdot z_{ab} \le 0 \text{ for all } b \in A\}.$$

The outcome function $f\colon \prod_{a\in A} S^a \to \prod_{a\in A}\mathbb{R}^\ell$ is defined as follows: Let $(p, z) = (p_a, (z_{ab})_{b\in A})_{a\in A}$ be a strategy profile and consider $a \in A$, then

$$f_a(p,z) = \sum_{b\in A} z_{ab} - \sum_{b\in A} z_{ba} + e(a).$$

Hence, each agent accepts all net trades which the other agents choose in his market and gets all the net trades he himself has chosen. Clearly, for all $(p, z) \in \prod_{a\in A} S^a$, $f(p, z) \in X(\mathcal{E})$.

Assume $|A| \ge 3$. It is easy to see that any Nash equilibrium of the generalized game is a Walrasian equilibrium. Indeed if $(p_a, (z_{ab})_{b\in A})_{a\in A}$ is a Nash equilibrium, then as $|A| \ge 3$, the absence of arbitrage possibilities implies that $p_a = p_b = p$ for all $a, b \in A$. Moreover as all agents can obtain their Walrasian demands at p, we have that the outcome corresponding to the Nash equilibrium is a Walrasian allocation.

One may, however, ask if one can define a non-generalized non-cooperative game such that the set of Nash equilibria in the game equals the set of Walrasian equilibria in the economy. One such result is Schmeidler (1980). Consider a finite economy $\mathcal{E} : A \to \mathbb{R}^\ell_+ \times \mathcal{P}^*_{mo}$. Define the strategy set S_a for agent a as

$$S_a = \{(p_a, z_a) \in \Delta \times \mathbb{R}^\ell \mid p_a > 0, \text{ and } p_a \cdot z_a = 0\}.$$

Let the outcome function $f\colon \prod_{a\in A} S^a \to \prod_{a\in A}\mathbb{R}^\ell$ be defined in the following way: Let $(p, z) = (p_a, z_a)_{a\in A}$ be a strategy profile and consider an agent $a \in A$. Let $T_a = \{b \in A \mid p_b = p_a\}$ and let

$$f_a(p,z) = \left(z_a - \frac{1}{|T_a|}\sum_{b\in T_a} z_b \right) + e(a).$$

Hence, all agents who have chosen the same price system are grouped and they obtain their chosen net trade adjusted by the average net trade in the group. Clearly, the outcome function f satisfies $\sum_{a\in A} f_a(p, z) = \sum_{a\in A} e(a)$ for all strategy profiles $(p, z) = (p_a,$

$z_a)_{a\in A}$.

Schmeidler (1980) shows the following result:

Theorem 10 **(Schmeidler)**

Let $\mathcal{E}$ be a finite economy in which the consumers have smooth preferences in $\mathcal{P}^*_{mo}$.

[(*i*)]*Assume that* (p, x) *is a Walrasian equilibrium for* $\mathcal{E}$. *Then* $(p, x(a) - e(a))_{a\in A} \in \prod_{a\in A} S_a$ *is a Nash equilibrium of the game described above. Assume* $|A| \geq 3$ *and that* $(p_a, z_a)_{a\in A}$ *is a Nash equilibrium of the game above. Then* $p_a = p_b = p$ *for all agents* $a, b \in A$ *and* $(p, (z_a + e(a))_{a\in A})$ *is a Walrasian Equilibrium for* $\mathcal{E}$.

The proof of assertion (ii) in the theorem is shown with the following steps: Let $(p_a, z_a)_{a\in A}$ be a Nash equilibrium. First, any agent a can obtain their Walrasian demand at any of the price systems p_b which are suggested by the other agents. Second, if a price system is chosen by at least two agents all agents choose it. Third, as there are more than two consumers and preferences are smooth, there cannot be a Nash equilibrium where all of the agents have chosen different price systems.

7. Conclusion

In this chapter we have reviewed some of the fundamental equivalence results for pure exchange economies. Clearly, it has not been possible to mention all the different results obtained in the literature.

We have defined an economy by using a measure space of agents and we have based the description of the economy on the individual agents' characteristics. However, in the literature there are several other ways of describing the basic economy. Vind (1964) models the basic exchange economy by a measurable space; however he bases the description of the economy on the characteristics of the coalitions, instead of the characteristics of the individual agents. There are also models with an uncountable number of agents that don't impose a measure space structure on the agents. Keiding (1974) gives Core equivalence results for economies with an abstract infinite index set of agents only allowing finite coalitions to improve upon allocations. Furthermore, we have not mentioned the large literature in which Core equivalence results are obtained by using non- standard analysis; see for example Brown and Robinson (1975).

Furthermore, in this chapter we have only considered pure exchange economies. However, the Core equivalence has been extended to economies with production. Indeed, endowing each coalition with a production set allows one to define the Core of a production economy, and again in atomless economies and with additivity assumptions on the production correspondence, Core equivalence has been obtained, see for example Hildenbrand (1974).

We briefly mentioned some results in which the Walrasian allocations are obtained from strategic behavior in non-cooperative games. In these examples the strategy space of the consumers are huge and the equivalence holds for a given finite economy. However, one might also consider sequences of economies with a growing number of agents. For each of the economies one can define a non-cooperative game and the Nash equilibria in this game. One might then investigate whether there is a connection between the set of Nash equilibria and the set of Walrasian allocations when the economy is sufficiently large. There are several papers along these lines. The paper by Gabszewicz and Vial (1972) is one of the first containing a general convergence result.

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Glossary

A Value Allocation:	It is assumed that agents' preferences are represented by utility functions. A Value allocation is an attainable allocation, where each agent gets a utility corresponding to his expected marginal contribution to the welfare of the groups in which he is member.
Atom:	In a measure space $(A, \mathcal{A}, \lambda)$ a set $S \in$ A is an atom if there does not exist $T \subset S$, $T \in \mathcal{A}$ such that $\lambda(T) > 0$ and $\lambda(S \setminus T) > 0$.
Atomless Economy:	An economy in which the set of agents is described by an atomless positive measure space.
Bargaining Set:	The set of attainable allocations in an economy, which cannot be improved upon with a justified objection.
Club Economy:	An economy where agents have the possibility to form groups and where membership in groups influence agents' preferences for consumption of private goods.
Core:	The set of attainable allocations for an economy, which cannot be improved upon.
Counter-objection:	A group of agents and an allocation for the group, which upsets an objection.
Generalized Game:	A non-cooperative game in which the set of feasible strategies for a player is allowed to vary with the players' strategy profile.
Improve Upon:	A coalition can improve upon an allocation if the members, by redistributing their initial endowments, can get consumption bundles they prefer.
Justified Objection:	An objection from a coalition which cannot be met with a counter-objection.
Lyapunov's Theorem:	States that the range of an atomless, non-negative, and finite vector valued measure is convex and compact.
Nash Equilibrium:	Solution concept for non-cooperative games. A strategy profile is a Nash equilibrium if all players use strategies that maximize their payoffs given the strategies of the other players.
Pure Exchange Economy:	A set of agents together with a description of their consumption sets, initial endowments, and preference relations.
Shapley-Folkman's	An approximate version of Lyapunov's Theorem.

Theorem:

Strongly Fair Net Trades: An allocation has strongly fair net trades if no agent prefers a combination of the net trades revealed by x with non-negative integer weights.

Walrasian Equilibrium: A pair consisting of a price system and an allocation. The allocation assigns to each agent an optimal bundle given the price system. Moreover all markets clear.

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Biographical Sketch

Birgit Grodal is Professor of Economics at University of Copenhagen. She received Master's in Mathematics 1968 and gold medal for a dissertation in mathematical economics 1970 from the University of Copenhagen. She has done extensive research within economic theory and especially in general equilibrium theory. In particular she has contributed to the theory of the core, economies with imperfect competition, and economies with incomplete markets. Moreover, she has contributed to the theory of economies with clubs. She has a large number of publications, including papers in *Econometrica*, *Review of Economic Studies*, *Journal of mathematical Economics*, and *Economic Theory*. She has been associated editor of *Econometrica* and *Economic Theory*, and has been on the editorial board of *Journal of Mathematical economics*. She is fellow in the Econometric Society, and has been on the Council and the Executive Committee of the Econometric Society. She is member of Academia Europeae. She has been visiting researcher at several American and European Universities.